[2][a] $\frac{dA}{dt} = (15)(4) - (20)\frac{A}{200-5t}$ $\quad A(0) = (200)(12)$

$$\underline{\frac{dA}{dt} = 60 - \frac{4A}{40-t}}, \quad \underline{A(0) = 2400}$$

3 4½ 2

$\underline{t \in [0, 40]}$ 3

TANK EMPTIES IN $\frac{200}{20-15} = 40$ MINUTES 3

AFTER THAT, SOLUTION CANNOT LEAVE AT 20 gal/min

[b] NO, DE IS NOT AUTONOMOUS 4½

[c] $\underline{\frac{dA}{dt} + \frac{4}{40-t}A = 60}$ 3 $\quad \mu = e^{\int \frac{4}{40-t}dt} = e^{-4\ln|40-t|} = \underline{(40-t)^{-4}}$ 4½

$$\underline{(40-t)^{-4}\frac{dA}{dt} + 4(40-t)^{-5}A = 60(40-t)^{-4}}$$ 3

CHECKPOINT: $\frac{d}{dt}(40-t)^{-4} = -4(40-t)^{-5}(-1)$

$= 4(40-t)^{-5}$ 3 ✓

$$(40-t)^{-4}A = \int 60(40-t)^{-4}\,dt + C$$

$u = 40-t$ 1½

$du = -dt$

$-\int 60u^{-4}\,du$ 1½

$$\underline{(40-t)^{-4}A = 20(40-t)^{-3} + C}$$ 3

$$A = 20(40-t) + C(40-t)^4$$

$$\underline{2400 = 20(40) + C(40)^4}$$ 3

$$2400 = 800 + C(40)^4$$

$$1600 = C(40)^4$$

$$C = \frac{1}{40^2} = \underline{\frac{1}{1600}}$$ 3

$$\underline{A(t) = 20(40-t) + \frac{1}{1600}(40-t)^4}$$ 3

[d] $\lim_{t \to 40^-} 20(40-t) + \frac{1}{1600}(40-t)^4 = \underline{0}$ 3

$\boxed{\text{EMPTY TANK, NO SALT}}$ 3

[e] $C(t) = \dfrac{A(t)}{200-5t} = \dfrac{20(40-t) + \frac{1}{1600}(40-t)^4}{5(40-t)}$

$= \underline{4 + \frac{1}{8000}(40-t)^3}$ $4\frac{1}{2}$

[f] $\lim_{t \to 40^-} C(t) = \underline{4}$ 3

$\boxed{\text{LAST DROP OF SOLUTION IN TANK CAME FROM BRINE FLOWING INTO TANK (WHICH HAD A CONCENTRATION OF 4 OUNCES SALT PER GALLON)}}$

3

[3] $(2w^3+t^3)\,dw+(w^3-wt^2)\,dt=0$ (4)

$2(kw)^3+(kt)^3=k^3(2w^3+t^3)$ (3)

$(kw)^3-(kw)(kt)^2=k^3(w^3-wt^2)$ (3)

BOTH HOMOGENEOUS ORDER 3

LET $t=vw \longrightarrow dt=v\,dw+w\,dv$ (4)

$(2w^3+v^3w^3)\,dw+(w^3-v^2w^3)(v\,dw+w\,dv)=0$ (4)

$(2+v^3)\,dw+(1-v^2)(v\,dw+w\,dv)=0$

$(2+\cancel{v^3}+v-\cancel{v^3})\,dw+w(1-v^2)\,dv=0$

$(2+v)\,dw+w(1-v^2)\,dv=0$ (6)

$\int \frac{1}{w}\,dw=\int \frac{v^2-1}{v+2}\,dv$ (4)

CHECKPOINT: SEPARABLE (3)

$\ln|w|+C=\int\left(v-2+\frac{3}{v+2}\right)dv$

$$\begin{array}{r|l} & v-2 \\ \hline v+2 & v^2 \qquad -1 \\ & \underline{v^2+2v} \\ & -2v-1 \\ & \underline{-2v-4} \\ & 3 \end{array}$$ (4)

(7) $\ln|w|+C=\frac{1}{2}v^2-2v+3\ln|v+2|$

(4) $\ln|w|+C=\frac{1}{2}\left(\frac{t}{w}\right)^2-2\left(\frac{t}{w}\right)+3\ln\left|\frac{t}{w}+2\right|$

$2w^2\ln|w|+Cw^2=t^2-4tw+6w^2(\ln|t+2w|-\ln|w|)$

(4) $Cw^2=t^2-4tw+6w^2\ln|t+2w|-8w^2\ln|w|$

ALTERNATE SOLUTION

[3] $(2w^3+t^3)\,dw + (w^3 - wt^2)\,dt = 0$ 4

$2(kw)^3+(kt)^3 = k^3(2w^3+t^3)$ 3

$(kw)^3-(kw)(kt)^2 = k^3(w^3-wt^2)$ 3

BOTH HOMOGENEOUS ORDER 3

LET $w = vt \to dw = t\,dv + v\,dt$ 4

$(2v^3t^3+t^3)(t\,dv+v\,dt)+(v^3t^3-vt^3)\,dt=0$ 4

$(2v^3+1)(t\,dv+v\,dt)+(v^3-v)\,dt=0$

$(2v^3+1)t\,dv+(2v^4\cancel{+v}+v^3\cancel{-v})\,dt=0$

$(2v^3+1)t\,dv+(2v^4+v^3)\,dt=0$ 6

$\int \frac{1}{t}\,dt = -\int \frac{2v^3+1}{2v^4+v^3}\,dv$ 4 CHECKPOINT: SEPARABLE 3

7

$$\frac{2v^3+1}{2v^4+v^3} = \frac{2v^3+1}{v^3(2v+1)} = \frac{\overset{4}{\cancel{A}}}{v} + \frac{\overset{-2}{\cancel{B}}}{v^2} + \frac{\overset{1}{\cancel{C}}}{v^3} + \frac{\overset{-6}{\cancel{D}}}{2v+1}$$

$2v^3+1 = Av^2(2v+1)+Bv(2v+1)+C(2v+1)+Dv^3$

$v=0$: $1 = C$

$v=-\frac{1}{2}$: $\frac{3}{4} = -\frac{1}{8}D \longrightarrow D=-6$

COEF OF $v^3$: $2 = 2A + D \longrightarrow A = \frac{1}{2}(2-D) = 4$

COEF OF $v^2$: $0 = A + 2B \longrightarrow B = -\frac{1}{2}A = -2$

SANITY CHECK $x=2$: $\frac{2(8)+1}{2(16)+8} = \frac{17}{40}$

$\frac{4}{2} - \frac{2}{4} + \frac{1}{8} - \frac{6}{5} = \frac{80-20+5-48}{40}$

$= \frac{17}{40}$ ✓

4 $\ln|t| = -\left(4\ln|v| + 2v^{-1} - \frac{1}{2}v^{-2} - 3\ln|2v+1|\right) + C$

4 $\ln|t| = -4\ln\left|\frac{w}{t}\right| - 2\left(\frac{t}{w}\right) + \frac{1}{2}\left(\frac{t}{w}\right)^2 + 3\ln\left|2\left(\frac{w}{t}\right)+1\right| + C$

$\cancel{\ln|t|} = -4\ln|w| + \cancel{4\ln|t|} - \frac{2t}{w} + \frac{t^2}{2w^2} + 3\ln|2w+t| - \cancel{3\ln|t|} + C$

$C(2w^2) = -8w^2\ln|w| - 4tw + t^2 + 6w^2\ln|2w+t|$

4 $Cw^2 = t^2 - 4tw + 6w^2\ln|2w+t| - 8w^2\ln|w|$

[4] $\underbrace{(1+\ln y)}_{M}dy+(\underbrace{4z-4y\ln y-9)}_{N}dz=0$ (3)

$M_z=0$ (1) $N_y=-4(1\ln y+y\cdot\frac{1}{y})=-4(\ln y+1)$ (3)

$\frac{N_y-M_z}{M}=\frac{-4(\ln y+1)}{1+\ln y}=-4$ FUNCTION OF ONLY $z$ (3)

$\mu=e^{\int-4\,dz}=e^{-4z}$ (4½)

$\underbrace{(1+\ln y)e^{-4z}}_{M}dy+\underbrace{(4z-4y\ln y-9)e^{-4z}}_{N}dz=0$ (3)

$M_z=-4(1+\ln y)e^{-4z}=N_y=-4(\ln y+1)e^{-4z}$ (3) CHECKPOINT: EXACT (3)

$F=\int(1+\ln y)e^{-4z}dy=e^{-4z}(y+y\ln y-y)+C(z)$

$F=(y\ln y)e^{-4z}+C(z)$ (4½)

$F_z=-4(y\ln y)e^{-4z}+C'(z)$ (3)

$=(4z-4y\ln y-9)e^{-4z}$

(3) $C'(z)=(4z-9)e^{-4z}$ CHECKPOINT: ONLY $z$ (3)

(3) $C(z)=(2-z)e^{-4z}$

(3) $(y\ln y)e^{-4z}+(2-z)e^{-4z}=C$ (3)

(3) $y\ln y-z+2=Ce^{4z}$

(6)

| $u$ | $dv$ |
|---|---|
| $4z-9$ | $e^{-4z}$ |
| $4$ | $-\frac{1}{4}e^{-4z}$ |
| $0$ | $\frac{1}{16}e^{-4z}$ |

$-\frac{1}{4}(4z-9)e^{-4z}-\frac{1}{4}e^{-4z}$

$=-\frac{1}{4}(4z-8)e^{-4z}$

$=(2-z)e^{-4z}$